ISSN 1512-1232

UDC 519.64

ON THE POSSIBILITY OF IMPROVING THE ERROR ESTIMATE OF THE APPROXIMATION OF AN INTEGRAL OPERATOR OF ONE CLASS

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Abstract

The issue of improving the error estimate of a numerical scheme constructed to solve a certain integral equation of the second kind is considered. The equation is obtained by regularizing the Shtaerman equation of the first kind, known in elasticity theory, which describes the contact problem of two elastic cylinders with almost equal radii. The integral part of this equation contains a kernel with a weak singularity, and thus the given equation of the second kind has some features, which in general, despite its Fredholm property, complicates the direct application of known methods of approximate solution to the equation. However, based on a relatively easy to implement modification of the quadrature method, it is possible to construct a fairly effective scheme for solving the corresponding equation and obtain an error estimate indicating the order of convergence, which is actually done in [1]. The numerical scheme proposed in this article differs somewhat from the schemes usually used for Fredholm equations and, taking into account a number of properties of the equation under consideration, allows one to justify the corresponding computational process in the space $H_{\frac{1}{2}}[-x_0, x_0]$ *with obtaining an error estimate of the form* $O(n^{-1/2})$. *The presented work will discuss the possibility of improving this estimate.*

Keywords: *Approximation of integral operator, numerical scheme, error estimation.*

In the work [1] a certain regularization process is considered, by means of which the Shtaerman equation of the first kind [2] known in the theory of elasticity is reduced to an equation of the second kind of Fredholm type. It also presents a scheme for an approximate solution of this equation, which is based on the approximation of the corresponding integral operator of the following form:

$$A\Gamma \equiv \int_{-x_0}^{x_0} K(x,t)\Gamma(t)dt,$$

where $\Gamma(t)$ is the desired function.

It should be noted here that the issues of approximate solution of such equations are currently developed quite well and, in particular, on the basis of using the quadrature method, it is possible to obtain effective a priori estimates. Despite this, in a given specific case, the use of known methods of numerical solution is still associated with certain difficulties. In particular, the structure of the kernel of the equation under consideration plays a significant role here

$$K(x,t)=\frac{\mu\sqrt{x_0^2-t^2}}{1+t^2}\int_0^x\frac{\cos[\gamma(x)-\gamma(s)]}{(t-s)\sqrt{x_0^2-s^2}}ds-$$
$$-\mu\int_0^x\frac{\cos[\gamma(x)-\gamma(s)]}{(1+s^2)(t-s)}ds. \quad (1)$$

where

$$\gamma(x)=\mu\,\mathrm{arctg}\,x\,,\quad x,t\in[-x_0,x_0],\quad \mu=\mathrm{Const},$$

and the value of the number x_0 is determined by the physical content of the problem under consideration.

Using the known results (see [3]) on the behavior of the singular integral near the ends of the integration interval, we can verify that the specified function has singularities on the interval $[-x_0,x_0]$.

Indeed, according to the mentioned results from [3], the first integral and, therefore, the first term as a whole in (1) is bounded as $t,x\to\pm x_0$.

As for the second integral in (1), it has a logarithmic singularity, as a result of which the kernel $K(x,t)$ is unbounded.

Consequently, despite the Fredholm property of the above-mentioned equation, direct application of known methods of approximate solution to it is impractical. However, based on a relatively easy-to-implement modification of the quadrature method, it is possible to construct a fairly effective scheme for solving the corresponding equation and obtain an error estimate indicating the order of convergence, which is actually done in the work under consideration [1]. The numerical scheme proposed there differs somewhat from the schemes usually used for Fredholm equations and, taking into account a number of properties of the equation under consideration, allows one to justify the corresponding computational process in the space $H_{\frac{1}{2}}[-x_0,x_0]$ with obtaining an error estimate of the form $O(n^{-1/2})$.

Here we come to the main goal of this work. We will show that, based on the properties of the sought function Γ, the above estimate of the approximation error of the integral operator $A\Gamma$ can be somewhat improved in a certain sense. In this case, our reasoning is based on the well-known approach of S.M. Nikolsky [4] to estimating the remainder of quadrature formulas of a complicated form.

So, let's consider (see [1])

$$\int_{-x_0}^{x_0}K(x,t)\Gamma(t)dt\approx\sum_{k=0}^{2n-1}L(\xi_k,\xi_{k+1},\Gamma), \quad (2)$$

$$\xi_j=-x_0+\frac{x_0}{n}j\ \ (j=0,1,\dots,2n),$$

where indicated

$$L(\xi_k,\xi_{k+1},\Gamma)=b_{k,n}^{(1)}(x)L(\xi_k)+b_{k,n}^{(2)}(x)L(\xi_{k+1}),$$

$$b_{k,n}^{(1)}(x)=\int_{\xi_k}^{\xi_{k+1}}\frac{t-\xi_{k+1}}{\xi_k-\xi_{k+1}}K(x,t)dt,$$

ISSN 1512-1232

$$b_{k,n}^{(2)}(x) = \int\limits_{\xi_k}^{\xi_{k+1}} \frac{t-\xi_k}{\xi_{k+1}-\xi_k} K(x,t)dt.$$

According to the proposed scheme, the calculation of singular integrals in the coefficients $b_{k,n}^{(1)}(x)$ and $b_{k,n}^{(2)}(x)$ is carried out using a properly selected approximate formula from [5] (see also [6]). To calculate the remaining integrals, one can use the rectangle formula with a certain weight function.

An estimate of the error of the corresponding approximation can be obtained by proceeding in a similar way to that known from the general theory of approximate methods (see [7]), which ultimately gives the estimate of the error indicated above.

According to [8], the function $\Gamma(x)$ is absolutely continuous and has a derivative summable on the interval $[-x_0, x_0]$

$$\Gamma'(x) = \frac{\Phi(x)}{\sqrt{x_0^2 - x^2}}\,, \tag{3}$$

where

$$\Phi(t) \in H_{\frac{1}{2}}[-x_0, x_0] \;\; (-x_0 < x < x_0).$$

Using this, as a result of a series of transformations, the remainder term of the quadrature formula (2) can be represented in integral form (compare with [4]):

$$\int\limits_{-x_0}^{x_0} K(x,t)\Gamma(t)dt - \sum_{k=0}^{2n-1} L(\xi_k, \xi_{k+1}, \Gamma) = \tag{4}$$

$$= \frac{x_0^2}{n^2} \sum_{k=0}^{2n-1} \int\limits_0^1 S_k(x,u)\Gamma'\left(\xi_k + \frac{x_0}{n}u\right) du,$$

where

$$\int\limits_0^1 S_k(x,u) = \int\limits_u^1 K\left(x, \xi_k + \frac{x_0}{n}y\right) dy -$$

$$-\int\limits_0^1 K\left(x, \xi_k + \frac{x_0}{n}y\right) ydy, \quad -x_0 < x < x_0.$$

Now, using representation (3), from (4) we find the estimate

$$\left| \int\limits_{-x_0}^{x_0} K(x,t)\Gamma(t)dt - \sum_{k=0}^{2n-1} L(\xi_k, \xi_{k+1}, \Gamma) \right| \leq$$

$$\leq \frac{x_0^2}{n^2} \max_{\substack{0\leq u\leq 1 \\ 0\leq k\leq 2n-1}} |S_k(x,u)| \sum_{k=0}^{2n-1} \int\limits_0^1 \left|\Gamma'\left(\xi_k + \frac{x_0}{n}u\right)\right| du = \tag{5}$$

$$= \frac{x_0}{n} \max_{\substack{0\leq u\leq 1 \\ 0\leq k\leq 2n-1}} |S_k(x,u)| \sum_{k=0}^{2n-1} \int\limits_{-x_0}^{x_0} |\Gamma'(t)|dt \leq \frac{M_1}{n} \max_{\substack{0\leq y\leq 1 \\ 0\leq k\leq 2n-1}} |S_k(x,y,)|,$$

ISSN 1512-1232

where

$$M_1 = x_0 \pi \max_{-x_0 \le t \le x_0} |\Phi(t)|.$$

Taking into account that, based on the result indicated in [8],

$$\max_{\substack{0 \le y \le 1 \\ 0 \le k \le 2n-1}} |S_k(x,y)| \le M_2,$$

$$M_2 = \text{Const}; \;\; k = 0,1,\dots,2n-1,$$

then from (5) we get

$$\left| \int_{-x_0}^{x_0} K(x,t)\Gamma(t)dt - \sum_{k=0}^{2n-1} L(\xi_k, \xi_{k+1}, \Gamma) \right| \le \frac{M}{n},$$

where M is a certain constant.

Acknowledgement

This research was supported by the European Union's Grant Project GAIN (grant agreement no.101078950)

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Article received: 2025-06-16