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EQUATIONS OF MOTION FOR SUPERFLUID $He^3 - He^4$ SOLUTIONS FILLED POROUS MEDIA

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ABSTRACT. The theory of the deformation of porous elastic solid containing a compressible superfluid He^4 has been considered in earlier publication. In the present paper, hypothetic experiments of measurement are described for the determination of the elastic coefficients of the theory. We aim at extending classical theory for the case when the porous media is saturated with superfluid $He^3 - He^4$ mixture. Finally, derived equations are applied to the most important particular case when the normal fluid component is locked inside a highly porous media by viscous forces. It is shown, that in highly porous media there exist two longitudinal sound modes: one is the intermediate mode between the first and fourth sound and another is the second sound like mode.

1. INTRODUCTION

We consider the case, when impurities participate only in normal fluid flow [1]. Sound propagation in a superfluid $He^3 - He^4$ solution has a number of peculiarities connected with the oscillation of the He^3 concentration in the acoustic wave. Whereas in pure helium II only the pressure oscillates in the first sound wave, and only the temperature oscillates in the second sound wave (neglecting the coefficient of thermal expansion, which is enormously small for helium), in a solution there are pressure, temperature, and concentration oscillations in both waves. In the first sound wave the oscillation of the temperature is proportional to the coefficient

$\beta = (c/\rho)\frac{\partial\rho}{\partial c}$ and in the second sound wave the same coefficient is proportional to the pressure oscillation (c- maximum He^3 concentration, ρ - density of the solution), and at low He^3 concentration the quantities proportional to β cannot be neglected (β = -0.3-0.4 for highly concentrated solutions).Unlike pure He^4, the first sound wave in solutions contains a relative oscillation of the normal and superfluid liquids, the magnitude of which is proportional to β. In pure He^4, there are no oscillations of the total flux $\vec{J} = \rho^n \vec{V}^n + \rho^S \vec{V}^S$ in the second sound wave, whereas in the solution the deviation from the equilibrium value of $\vec{J}$ is also proportional to β [1]. On the other hand when aerogel is saturated even with pure He II new phenomena are caused by the presence of aerogel: namely, the coupling between two sound modes is provided by $\sigma\rho^a\rho^S$ (ρ^a -is the aerogel density, σ - He^3 - He^4 solution entropy) [2]. So, in this paper we have considered the peculiarities of sound propagation for impure homogeneous superfluids, where these phenomena are caused by both impurities (including He^3 in He II) and by the presence of aerogel. The task of the article represents the derivation of hydrodynamic equations for consolidated porous media filled with superfluid $He^3 - He^4$ solution and determination of all input elastic coefficients of the theory by physically measured quantities without any additional adjustable parameters.

EXPRESSION OF GENERALIZED COEFFICIENTS BY PHYSICALLY MEASURED QUANTITIES

The elastic properties of a system containing a superfluid helium completely filling the pores were considered in [3], where methods for measurement of generalized elastic coefficients are described with jacketed and unjacketed compressibility tests in the case of a homogeneous and isotropic porous matrix. In our case according to [3, 4,5] the stress-strain relations are

$$\sigma_x = 2Ne_x + Ae + Q^S\varepsilon^S + Q^n\varepsilon^n,$$
$$\sigma_y = 2Ne_y + Ae + Q^S\varepsilon^S + Q^n\varepsilon^n,$$

$$\sigma_z = 2Ne_z + Ae + Q^S\varepsilon^S + Q^n\varepsilon^n,$$

$$\tau_x = N\gamma_x,\quad \tau_y = N\gamma_y,\quad \tau_z = N\gamma_z. \qquad (1)$$

$$s' = Q^S e + R^S\varepsilon^S + R^{sn}\varepsilon^n,$$

$$s'' = Q^n e + R^n\varepsilon^n + R^{sn}\varepsilon^S,$$

where $\sigma_x, \sigma_y, \sigma_z$ and τ_x, τ_y, τ_z are normal and tangential forces acting on the solid parts of each face of the cube with the following orientation, s' and s'' are forces acting on the solution part of each face of the cube corresponding to superfluid and normal components of superfluid solution. Scalars s' and s'' are expressed in the following form

$$s' = -\Phi p^S,\ s'' = -\Phi p^n. \qquad (2)$$

Here $p^S = \rho^S\mu$, $p^S + p^n = p$ [6], where μ is chemical potential, p - liquid pressure and Φ - porosity. Thus, we have taken into consideration the circumstance that the existence of pressure gradient is not enough for acceleration of superfluid and normal components of superfluid liquid unlike usual fluid.

The average displacement vector of the solid has the components u_x, u_y, u_z and that of the mixture $U_x^S, U_y^S, U_z^S, U_x^n, U_y^n, U_z^n$. The solid strain components are then given by

$$e_x = \frac{\partial u_x}{\partial x},\; e_y = \frac{\partial u_y}{\partial y},\; e_z = \frac{\partial u_z}{\partial y},$$

$$\gamma_x = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y},\;\; \gamma_y = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x},\;\; \gamma_z = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}. \tag{3}$$

Due to two possible types of motion in He II $\vec{U}$ breaks down into the sum of two parts

$$\vec{U} = \frac{\rho^S}{\rho}\vec{U}^S + \frac{\rho^n}{\rho}\vec{U}^n \tag{4}$$

corresponding to displacement of superfluid and normal components. Thus the strain in fluid is defined by the dilatation

$$\varepsilon = \frac{\rho^S}{\rho}\nabla\vec{U}^S + \frac{\rho^n}{\rho}\nabla\vec{U}^n . \tag{5}$$

Because of the fact that superfluid and normal part of He II cannot be divided physically and there is no sense to speak about belonging of some atoms to superfluid or normal components, the following relation is to be fulfilled

$$Q^S\varepsilon^S + Q^n\varepsilon^n = Q\varepsilon . \tag{6}$$

The coefficients A and N correspond to the well-known Lame coefficients in the theory of elasticity and are positive. The coefficients Q and R are the familiar Biot's coefficients [7]. The physical interpretation of the coefficients R^S, R^n, R^{sn} is given in [1,5].

To illustrate the above mentioned let us discuss some cases of experiments which may be used to relate generalized elastic coefficients of the theory to the directly measurable coefficients: the bulk modulus of fluid K_f, the bulk modulus of solid K_{sol}, the bulk modulus of the skeletal frame K_b and N.

For clear determination we note, that in the unjacketed compressibility experiment, a sample of the porous solid is immersed in a superfluid He^3-He^4 solution to which a pressure p' is applied. Under the action of pressure the solution penetrates the pores completely and the dilations of the porous solid e and solutions ε are measured. Unjacketed elastic coefficients of solid and fluid are determined by

$$\frac{1}{K_{sol}}=-\frac{e}{p'}; \qquad \frac{1\varepsilon}{K_f}=-\frac{}{p'} \tag{7}$$

Also we note, that from expression of the solution chemical potential we have the form of the force acting on the superfluid component and normal component portions:

$$\sigma^S=-\Phi\frac{\rho^S}{\rho}\left(1+\beta\right)p';$$

$$\sigma^n=-\Phi\frac{\rho^n}{\rho}\left(1-\frac{\rho^S}{\rho^n}\beta\right)p'. \tag{8}$$

After considering these conditions $\varepsilon^{\mathbf{s}}=\varepsilon^{\mathbf{n}}=\varepsilon$ we have:

$$\left(\frac{2}{3}N+A\right)\frac{1}{K_{sol}}+\left(Q^S+Q^n\right)\frac{1}{K_f}=\left(1-\Phi\right),$$

$$Q^S \frac{1}{K_{sol}} + \left(R^S + R^{sn}\right) \frac{1}{K_f} = \Phi \frac{\rho^S}{\rho} \left(1 + \beta\right), \tag{9}$$

$$Q^n \frac{1}{K_{sol}} + \left(R^n + R^{sn}\right) \frac{1}{K_f} = \Phi \frac{\rho^n}{\rho} \left(1 - \frac{\rho^S}{\rho^n} \beta\right).$$

The following test corresponds to the jacketed compressibility test, when a specimen of the material is enclosed in a thin impermeable jacket and then subjected to an external fluid pressure p'. The dilatation of the specimen is measured and coefficient of jacketed compressibility K_b is determined by

$$\frac{1}{K_b} = -\frac{e}{p'} \tag{10}$$

and also we have relations

$$\sigma_x = \sigma_y = \sigma_z = -p'; \qquad \varepsilon^S = \varepsilon^n = \varepsilon; \qquad s' = s'' = 0. \tag{11}$$

Therefore we have the three relations

$$\left(\frac{2}{3} N + A\right) e + \left(Q^S + Q^n\right) \varepsilon = -p',$$

$$Q^S e + \left(R^S + R^{sn}\right) \varepsilon = 0, \tag{12}$$

$$Q^n e + \left(R^n + R^{sn}\right) \varepsilon = 0.$$

From (9) and (12) it follows

$$Q=\frac{\Phi K_{sol}\left(1-\Phi-\dfrac{K_b}{K_{sol}}\right)}{1-\Phi+\Phi\dfrac{K_{sol}}{K_f}-\dfrac{K_b}{K_{sol}}}\ , \tag{13}$$

$$\frac{2}{3}N+A=K_{sol}\frac{(1-\Phi)\left(1-\Phi-\dfrac{K_b}{K_{sol}}\right)+\Phi\dfrac{K_{sol}}{K_f}}{1-\Phi+\Phi\dfrac{K_{sol}}{K_f}-\dfrac{K_b}{K_{sol}}}\ , \tag{14}$$

$$R^{n}+R^{sn}=K_f\frac{\rho^{n}}{\rho}\left(\Phi-\frac{Q}{K_{sol}}\right)\left(1-\frac{\rho^{S}}{\rho^{n}}\beta\right), \tag{15}$$

$$R^{S}+R^{sn}=K_f\frac{\rho^{S}}{\rho}\left(\Phi-\frac{Q}{K_{sol}}\right)(1+\beta)\,. \tag{16}$$

Let us consider the situation when the jacket is communicated with reservoir by the superleak. Therefore only superfluid component pours into reservoir and we can write the following relations:

$$\left(\frac{2}{3}N+A\right)e+Q^{S}\varepsilon^{S}+Q^{n}\varepsilon^{n}=-(1-\Phi)\,p'\,,$$

$$Q^{S}e+R^{S}\varepsilon^{S}+R^{sn}\varepsilon^{n}=0\,, \tag{17}$$

$$Q^{n}e+R^{n}\varepsilon^{n}+R^{sn}\varepsilon^{S}=-\Phi p'\,.$$

In this compressibility test we have not the relation between ε^{S} and ε^{n}. For its determination we should utilize the conservation laws of mass and entropy. Then we have

$$\left(\varepsilon^{s}-\varepsilon^{n}\right)\frac{\tilde{\sigma}^{2}}{\frac{\partial\sigma}{\partial T}}\frac{\rho^{s}}{\rho}=\frac{1+\beta}{\rho}p', \tag{18}$$

where

$$\tilde{\sigma}^{2}=\bar{\sigma}^{2}+c^{2}\frac{\partial}{\partial c}\left(\frac{Z}{\rho}\right)\frac{\partial\sigma}{\partial T};\qquad \bar{\sigma}=\sigma-c\frac{\partial\sigma}{\partial c}.$$

The quantity $Z=\rho\left(\mu_3-\mu_4\right)$ is defined in terms of the chemical potentials μ_3,μ_4 for He^3 and He^4 in the solution.

These equations (17-18) together with (13-16) give:

$$R^{sn}=\frac{\rho^{s}\rho^{n}}{\rho^{2}}\left(1+\beta\right)\left(1-\frac{\rho^{s}}{\rho^{n}}\beta\right)R-\frac{\left(\rho^{s}\right)^{2}\tilde{\sigma}^{2}T\Phi}{\rho\, C_{He}}, \tag{19}$$

$$R^{n}=\frac{\left(\rho^{n}\right)^{2}}{\rho^{2}}\left(1-\frac{\rho^{s}}{\rho^{n}}\beta\right)^{2}R+\frac{\left(\rho^{s}\right)^{2}\tilde{\sigma}^{2}T\Phi}{\rho\, C_{He}}, \tag{20}$$

$$R^{s}=\frac{\left(\rho^{s}\right)^{2}}{\rho^{2}}\left(1+\beta\right)^{2}R+\frac{\left(\rho^{s}\right)^{2}\tilde{\sigma}^{2}T\Phi}{\rho\, C_{He}}. \tag{21}$$

Where Biot-Willis coefficient R is equal to [7]

$$R=\frac{\Phi^{2}K_{sol}}{1-\Phi+\Phi\frac{K_{sol}}{K_{f}}-\frac{K_{b}}{K_{sol}}} \tag{22}$$

and C_{He} is the specific heat of the solution.

Equations for elastic waves are received by analogy with articles [1,5], expressing stress tensor through strain tensor. For three-dimensional cases we can write:

$$N\nabla^{2}\vec{u}+(A+N)\operatorname{grad}e+Q^{S}\operatorname{grad}\varepsilon^{S}+Q^{n}\operatorname{grad}\varepsilon^{n}=$$

$$=\frac{\partial^{2}}{\partial t^{2}}\left(\rho_{11}\vec{u}+\rho_{12}^{S}\vec{U}^{S}+\rho_{12}^{n}\vec{U}^{n}\right)+bF(w)\frac{\partial}{\partial t}\left(\vec{u}-\vec{U}^{n}\right),$$

$$Q^{S}\operatorname{grad}e+R^{S}\operatorname{grad}\varepsilon^{S}+R^{sn}\operatorname{grad}\varepsilon^{n}=\frac{\partial^{2}}{\partial t^{2}}\left(\rho_{12}^{S}\vec{u}+\rho_{22}^{S}\vec{U}^{S}\right),$$

$$Q^{n}\operatorname{grad}e+R^{n}\operatorname{grad}\varepsilon^{n}+R^{sn}\operatorname{grad}\varepsilon^{S}=$$

$$=\frac{\partial^{2}}{\partial t^{2}}\left(\rho_{12}^{n}\vec{u}+\rho_{22}^{n}\vec{U}^{n}\right)-bF(w)\frac{\partial}{\partial t}\left(\vec{u}-\vec{U}^{n}\right), \tag{23}$$

where ρ_{11} is total effective density of the solid moving in the He^{3} - He^{4} solution. Coefficients ρ_{12}^{S} and ρ_{12}^{n} are mass parameters of "coupling" between a solid and correspondingly, superfluid and normal components of solution or mass coefficient $\rho_{12}^{S(n)}$ describes the inertial (as opposed to viscous) drag that the fluid exerts on the solid as the latter is accelerated relative to the former and vice-versa.

It is well known, that the He^{3} - He^{4} solution densities have the following form [1,5]:

$$\rho_{22}^{s} = \Phi\rho^{s} - \rho_{12}^{n} - \rho_{12}^{s}\,; \qquad \rho_{22}^{n} = \Phi\rho^{n} - \rho_{12}^{n} - \rho_{12}^{s}$$

$$-\rho_{12}^{s} > 0\,, \qquad -\rho_{12}^{n} > 0\,.$$

Complex quantity F(w) describes the deviation from Poiseille flow at finite frequencies. The coefficient $b = \eta\Phi^{2}/k_{0}$ is the ratio of total friction force to the average normal fluid velocity, where η is the fluid viscosity and k_0 is the permeability.

SOUND PROPAGATION IN UNRESTRICTED GEOMETRY AND AEROGEL

Now it will be interesting to ignore dissipative process in equations (20) and consider the case of unrestricted geometry. Then from equations (23) we have

$$R^{S}\,\mathrm{grad}\,\varepsilon^{S} + R^{sn}\,\mathrm{grad}\,\varepsilon^{n} = \rho^{S}\frac{\partial^{2}\vec{U}^{S}}{\partial t^{2}}\,,$$

$$R^{n}\,\mathrm{grad}\,\varepsilon^{n} + R^{sn}\,\mathrm{grad}\,\varepsilon^{S} = \rho^{n}\frac{\partial^{2}\vec{U}^{n}}{\partial t^{2}}\,. \qquad (24)$$

Here we take into account that in the limit interest to us purely geometrical quantity $\alpha_{¥}$, which is independently of solid or fluid densities, and porosity Φ are equal to one. Because the induced mass tensor per unit volume $\rho_{12}^{s(n)} = -(\alpha_{\infty} - 1)\,\Phi\rho^{s(n)}$ and

$$
\begin{aligned}
R^{S} &= \frac{\left(\rho^{S}\right)^{2}}{\rho^{2}}(1+\beta)^{2} K_{f} + \frac{\left(\rho^{S}\right)^{2}\sigma^{2}T}{\rho\, C_{He}} \\
R^{n} &= \frac{\left(\rho^{n}\right)^{2}}{\rho^{2}}\left(1-\frac{\rho^{S}}{\rho^{n}}\beta\right)^{2} K_{f} + \frac{\left(\rho^{S}\right)^{2}\sigma^{2}T}{\rho\, C_{He}} \qquad (25)\\
R^{sn} &= \frac{\rho^{S}\rho^{n}}{\rho^{2}}(1+\beta)\left(1-\frac{\rho^{S}}{\rho^{n}}\beta\right)K_{f} - \frac{\left(\rho^{S}\right)^{2}}{\rho C_{He}}\sigma^{2}T
\end{aligned}
$$

So, for pure He^3 - He^4 solution solving the system (24) in the usual manner we obtain the dispersion equation for the bulk waves propagating in free He^3 - He^4 solution:

$$
C^{4}\rho^{S}\rho^{n} - C^{2}\left(\rho^{S}R^{n} + \rho^{n}R^{S}\right) + R^{S}R^{n} - \left(R^{sn}\right)^{2} = 0 \qquad (26)
$$

Equation (26) has two roots

$$
C_1^2 = \frac{K_f}{\rho}\left(1+\frac{\rho^{S}}{\rho^{n}}\beta^{2}\right); \qquad C_2^2 = \frac{\rho^{S}}{\rho^{n}}\frac{\sigma^{2}T}{C_{He}\left(1+\frac{\rho^{S}}{\rho^{n}}\beta^{2}\right)} \qquad (27)
$$

which conform to the velocity of the first and the second sounds correspondingly [8].

From (24) equations it follows the well known results for the fourth sound in free He^3 - He^4 solutions [3]. If we assume $\vec{U}^{n}=0$ in (21), we derive [9,10]

$$
C_4^2 = \frac{\rho^{n}}{\rho}C_1^2\frac{(1+\beta)^{2}}{1+\frac{\rho^{S}}{\rho^{n}}\beta^{2}} + \frac{\rho^{n}}{\rho}C_2^2\left(1+\frac{\rho^{S}}{\rho^{n}}\beta^{2}\right). \qquad (28)
$$

Propagation of the fourth sound in a He^3 - He^4 solution was studied in [9,10] from the hydrodynamic equations.

A great deal of effort has recently been dedicated to the investigation of superfluid solution in porous materials. We cite here recent articles describing the specific features of superfluid liquid in various porous structures [11]. The sound velocity in porous media can provide information about the superfluidity property as well as elastic properties of the solid matrix. McKenna et al [12] developed a theory explaining the behavior of sound modes in aerogel filled with He II, taking into account coupling between the normal component and the aerogel and its elasticity. Here the normal component is locked in a very compliant solid matrix so that the liquid and aerogel fibers move together under mechanical and thermal gradients. It takes place at low sound frequencies, when the viscous penetration depth in bigger than the pore size so the entire normal component is viscously locked to the solid matrix. In this case from (23) for longitudinal waves we have the following dispersion equation:

$$\rho^S[\rho^a+\rho^n)]C^4-C^2[R^S(\rho^a+\rho)+\rho^S(A+2N+2Q+R)-2\rho^S\times$$
$$\times(Q^S+R^S+R^{sn})]+R^S(A+2N+2Q+R)-(Q^S+R^S+R^{sn})^2=0 \quad (29)$$

The bulk velocities can express this dispersion equation:

$$\left(1+\frac{\rho^a}{\rho^n}\right)C^4-C^2\{C_1^2+\left(1+\frac{\rho^S}{\rho^n}\beta^2\right)C_2^2+\frac{\rho^a}{\rho^n}\left(C_a^2+C_4^2\right)\}+$$
$$+C_1^2C_2^2+\frac{\rho^a}{\rho^n}C_a^2C_4^2=0 \quad (30)$$

here $C_a^2=\dfrac{K_b+(4/3)N}{\rho^a}$.

The first solution is intermediate between the first and fourth sound

$$C_{14}^2=\frac{C_1^2+\frac{\rho^a}{\rho}C_a^2}{1+\frac{\rho^a}{\rho^n}} \qquad (31)$$

and it resembles the fast mode.

Another solution corresponds to the slow mode, which is an oscillation of a deformation of the aerogel combined with a simultaneous out-of-phase motion of the superfluid component:

$$C_{2a}^2=\frac{C_2^2+\frac{\rho^a\rho^S}{\rho^n\rho}C_4^2C_a^2}{1+\frac{\rho^a\rho^S}{\rho^n\rho}}. \qquad (32)$$

In this wave the main oscillated quantity is temperature. From experiment data for silica aerogel $C_a^2 >> C_2^2$ [12], so from the above mentioned formula it follows that $C_{2a}^2 >> C_2^2$ Therefore, the velocity of slow wave is much bigger than that of temperature sound in free solutions.

From (31) and (32) it follows that an aerogel filled with superfluid He^3 - He^4 solution simultaneously possesses the properties of elastic solid and superfluid liquid. Also, in this paper we have considered the peculiarities of sound propagation for impure homogeneous superfluids, where these phenomena are caused both by impurities (including He^3 in He II) and by the presence of aerogel.

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He^3 - He^4 ხსნარით შევსებულ ფოროვან გარემოს მოძრაობის განტოლებები

დასკვნა

სტატიაში მიღებულია He^3 - He^4 ხსნარით შევსებული ფოროვან გარემოს მოძრაობის გაწრფივებული განტოლებები. მათში შემავალი განზოგადებული დრეკადობის კოეფიციენტები გამოსახულია ექსპერიმენტულად გაზომვადი ფიზიკური შინაარსის მქონე სიდიდეებით. მიღებული განტოლებების გამოყენებით განსაზღვრულია

მაღალი ფოროვნების მქონე გარემოში - აეროგელში გავრცელებადი გრძივი ბგერების სიჩქარეები. ნაჩვენებია, რომ სწრაფი ბგერის სიჩქარე წარმოადგენს პირველი და მეოთხე ბგერების სიჩქარეების კომბინაციას, ხოლო ნელი ბგერა ეთანადება ტემპერატურულ ბგერას, რომლის სიჩქარე სილიკა აეროგელებში მეტია თავისუფალ ხსნარებში ტემპერატურულ ბგერის სიჩქარესთან შედარებით.