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# MODIFIED HYPERSPHERICAL FUNCTION METHOD FOR 3 PARTICLES SYSTEM IN 2D SPACE WITH INVERSE SQUARE POTENTIAL BETWEEN PARTICLES 

A. Lomidze, Sh.Tsiklauri

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#### Abstract

The present investigation reports, that modified hyperspherical function method permits us to advance comparatively simple and nonmodel representation of the solution of three particles problem in 2D space for inverse square of pair interaction. This becomes possible due to the correlation function along with the effective potential there appears $r^{-1}$ potential7 leading the ground-state energy to the finite quantity. In a first approximation the problem is decided analytically.


Numerous physical phenomena may be described by singular potentials [1]. Inverse square potential is more interesting, especially in polymers [2], and in the interaction between Rydberg atom and polar molecule [3]. The problem of three particles by pair interaction of $\frac{\mathrm{g}}{\mathrm{r}^{2}}$ type in 1D space in a field harmonic oscillator has been worked out analytically [4]. The problem of three particles for pair inverse square interaction in 3D space has been studied [5], the same system in 2D space is considered in the present paper.

Schrodinger equation in a system of center of masses for three nonidentical and non relativistic particles in hyperspherical coordinates in 2D space are defined in [6]. When we solve Schrodinger equation we get coupled differential equations system for hyperradial function. We consider only one equation from this system, when $\mathrm{K}=\mathrm{K}$ ' ( K is hypermoment of the particles). As a result the following equation have been gotten:

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$$
\begin{equation*}
\left(\frac{\partial^{2}}{\partial \rho^{2}}-\left[\chi^{2}+\frac{(\mathrm{K}+1)^{2}-0,25}{\rho^{2}}\right]\right) \varphi_{\mathrm{K}}(\rho)=\frac{2 \mu}{\rho^{2}} \mathrm{~J}_{\mathrm{K}} \varphi_{\mathrm{K}}(\rho) \tag{1}
\end{equation*}
$$

where:

$$
\begin{equation*}
\mathrm{J}_{\mathrm{K}}=\int \Phi_{\mathrm{K}}^{*}(\Omega) \Phi_{\mathrm{K}}(\Omega)(\cos \alpha)^{-2} \mathrm{~d} \Omega \tag{2}
\end{equation*}
$$

and $\chi^{2}=-2 \mu \mathrm{E} ; \Phi(\Omega)$ is eigenfunction of the generalized angularmomentum operator which analytical form is known [6],

$$
\begin{equation*}
\Phi_{\mathrm{K}}(\Omega)=\mathrm{C}_{\mathrm{o}} \cos \alpha \sin \alpha \mathrm{P}_{\mathrm{n}}^{1,1}\left(\cos ^{2} \alpha\right) \tag{3}
\end{equation*}
$$

where:

$$
\mathrm{n}=\frac{\mathrm{K}}{2}, \quad \mathrm{C}_{\mathrm{o}}=\left[\frac{2(1 \cdot 2 \cdot \ldots \cdot \mathrm{n}) \Gamma(1+\mathrm{n})(1+2 \mathrm{n})}{\Gamma(1+\mathrm{n}) \Gamma(1+\mathrm{n})}\right]^{1 / 2}
$$

$\Omega \equiv(\alpha, \stackrel{\mathrm{r}}{\mathrm{x}}, \stackrel{\mathrm{r}}{\mathrm{y}})$ denotes five angles, $\rho$ and $\alpha$ are defined by the expression: $\rho^{2}=x^{2}+y^{2} ; \quad x^{\prime}$ and $y^{\prime}$ are the Jacoby coordinates; $|\mathrm{x}|=\rho \cos \alpha ; \quad|\mathrm{y}|=\rho \sin \alpha ;\left(0<\rho<\infty ; 0<\alpha<\frac{\pi}{2}\right) . \mathrm{E}<0$ is a binding energy for three particles; $\mu$ is a reduced mass. The integral (2) can be calculated analytically [6] and it is equal to

$$
\begin{array}{r}
\mathrm{J}_{\mathrm{K}}=2^{-(\mathrm{K}+1)} \frac{\mathrm{g}_{\mathrm{ij}} \mathrm{~m}_{\mathrm{i}} \mathrm{~m}_{\mathrm{j}}}{\left(\mathrm{~m}_{\mathrm{i}}+\mathrm{m}_{\mathrm{j}}\right) \mu_{\mathrm{k}=1}} \sum_{\mathrm{n}}\left[\binom{\mathrm{n}+1}{\mathrm{k}}\binom{\mathrm{n}+1}{\mathrm{k}+1}\right]^{2} \mathrm{~B}\left(\frac{3}{2}, \mathrm{~K}-2 \mathrm{k}+1\right) \times \\
 \tag{4}\\
x \frac{\Gamma(\mathrm{~K}+2)}{\Gamma(2 \mathrm{k}+1) \Gamma(\mathrm{K}-2 \mathrm{k}+2)}
\end{array}
$$

Simple analysis shows that under these conditions the ground state energy has infinitely large negative value and therefore there is

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nosense to solve it. To avoid this we use modified hyperspherical function method (MHFM) [7].

The main idea of the MHFM is that wave function $\Psi$ represents the product of two functions, where the first is the main hyperspherical function and the second is the correlation function$\zeta=\exp (f)$, defined by singularity and clustering properties of the wave function and it is equal to,

$$
\begin{equation*}
\mathrm{f}=-\sum_{\mathrm{i}=1}^{3} \gamma_{\mathrm{i}} \mathrm{r}_{\mathrm{i}} \tag{5}
\end{equation*}
$$

where $\mathrm{r}_{\mathrm{i}}$ is a distance between the particles and $\gamma_{i}$ is determined according to physical considerations.

Taking into account the relation between three different sets of Jacobi coordinates [8], the expression (5) can be presented as:

$$
\sum_{\mathrm{i}=1}^{3} \gamma_{\mathrm{i}} \mathrm{z}_{\mathrm{i}}=\rho\left(\mathrm{G}_{1} \cos \alpha+\mathrm{G}_{2} \sin \alpha\right)
$$

where:

$$
\begin{gathered}
\mathrm{G}_{1}=\gamma_{1}+\gamma_{2} \cos \left(\phi_{23}+\phi_{31}\right)-\gamma_{3} \cos \phi_{31} ; \\
\mathrm{G}_{2}=\gamma_{2} \sin \left(\phi_{23}+\phi_{31}\right)-\gamma_{3} \sin \phi_{31} .
\end{gathered}
$$

If we substitute expression (5') into (1), and carry out some transformations, hyperradial differential equation is obtained but we consider only one equation when $K=K^{\prime}$ :

$$
\begin{align*}
\left(\frac{\partial^{2}}{\partial \rho^{2}}+\left(\frac{3}{\rho}-\mathrm{W}_{2}^{\prime}\right)\right. & \frac{\partial}{\partial \rho}+\frac{3 \mathrm{~W}_{2}^{\prime}+\mathrm{W}_{3}^{\prime}}{\rho}+ \\
& \left.+\left(\chi^{2}+\mathrm{W}_{6}^{\prime}\right)-\frac{2 \mu}{\mathrm{~h}^{2}} \frac{\mathrm{~K}(\mathrm{~K}+2)+\mathrm{J}_{0}}{\rho^{2}}\right) \psi(\rho)=0 \tag{6}
\end{align*}
$$

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where:

$$
\begin{gathered}
\mathrm{W}_{2}^{\prime}=\left(\mathrm{G}_{1}-\mathrm{G}_{2}\right) \times \frac{21 \sqrt{6}}{8} ; \quad \mathrm{W}_{6}^{\prime}=\mathrm{G}_{1}^{2}+\mathrm{G}_{2}^{2} \\
\mathrm{~W}_{3}^{\prime}=\mathrm{W}_{2}^{\prime}+21 \sqrt{6}\left(0,25 \mathrm{G}_{1}+\mathrm{G}_{2}\right)
\end{gathered}
$$

Taking into account the asymptotic behavior of the equation (6), let us seek a solution as the following:

$$
\begin{equation*}
\psi(\rho)=\exp (-\delta \rho) \rho^{\sigma} \varphi(\rho) \tag{7}
\end{equation*}
$$

where:

$$
\sigma=-1+\sqrt{9+2 \mathrm{~mJ}_{\mathrm{o}}} ; \delta=\frac{\sqrt{\mathrm{W}_{2}^{\prime 2}-4\left(\chi^{2}+\mathrm{W}_{6}^{\prime}\right)}-\mathrm{W}_{2}^{\prime}}{4}
$$

Substituting expression (7) into equation (6), then for $\varphi(\rho)$ we obtain the equation of hypergeometrical function:

$$
\begin{equation*}
\left(\mathrm{r} \frac{\partial^{2}}{\partial \mathrm{r}^{2}}+(-\mathrm{r}+2 \sigma+3) \frac{\partial}{\partial \mathrm{r}}+\frac{\left(3 \mathrm{~W}_{2}^{\prime}+\mathrm{W}_{3}^{\prime}-3 \delta\right)}{2 \delta+\mathrm{W}_{2}^{\prime}}-\sigma\right) \varphi(\mathrm{r})=0 \tag{8}
\end{equation*}
$$

where: $\mathrm{r}=\left(2 \delta+\mathrm{W}_{2}^{\prime}\right) \rho$.
If we take into account that three body system is binded then solution of the (8) equation is represented as the following type of hypergeometrical function:

$$
\begin{equation*}
\varphi(\rho)=\mathrm{C}_{1} \mathrm{~F}(\mathrm{a}, \mathrm{~b}, \mathrm{r}) \tag{9}
\end{equation*}
$$

where: $\quad \mathrm{b}=2 \sigma+3 ; \quad \mathrm{a}=\frac{\left(3 \mathrm{~W}_{2}^{\prime}+\mathrm{W}_{3}^{\prime}-3 \delta\right)}{2 \delta+\mathrm{W}_{2}^{\prime}}=-\mathrm{N} ; \quad \mathrm{N}=1,2 \ldots$.

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For binding energy we received:

$$
\begin{equation*}
\mathrm{E}_{\mathrm{N}}=-\frac{\mathrm{h}^{2}}{8 \mu}\left[\left(\frac{12 \mathrm{~W}_{2}^{\prime}+4 \mathrm{~W}_{3}^{\prime}-\mathrm{W}_{2}^{\prime}(3+2 \sigma-2 \mathrm{~N})}{3+2 \sigma-2 \mathrm{~N}}\right)^{2}-\mathrm{W}_{2}^{2}+4 \mathrm{~W}_{6}^{\prime}\right] \tag{10}
\end{equation*}
$$

The dependence of binding energy of three body system on the global quantum number N obtained as a solution results in the expression (10) that is given in the Table.

Table. Dependence of the binding energy of the three body system in 2 D space upon the global quantum number N

| Global quantum <br> number N | Binding energy <br> -E (a.u.) |
| :---: | :---: |
| 1 | 0.114174 |
| 2 | 0.074499 |
| 3 | 0.005834 |
| 4 | 0.001928 |
| 5 | 0.001256 |
| 6 | 0.001143 |

(In these results we assume that correlation parameters and interaction constant are the same for all particles. Their variation doesn't give any qualitatively new results).

Thus MHFM permits us to advance comparatively simple and nonmodel representation of the solution of three particles problem in 2D space for inverse square of pair interaction. This is possible due to the correlation function together with effective potential appears to be 1 $\frac{1}{r}$ potential7 leading the ground-state energy to finite quantity. In a r first approximation the problem is decided analytically.

## REFERENCES

## Georgian Electronic Scientific Journals: Physics \#1(38-2)-2003

1. W.M.Frank, D.Land, R.M.Spector. Rev. Mod.Phys. 43, 1971, 36;
D. Amin. Phys. Today 35, 1982, 35; Phys. Rev. Lett. 36, 1976, 323; A. Khare, S. N. Behra, Pramana. J. Phys. 14, 1980, 327; S. Colemann. Aspects of Symmetry selected Erice Lectures, Cambridge, 1988.
2. E.Marinari, G.Parisi. Europhys. Lett. 15, 1991, 721.
3. C.Desfranqois, H. Abdoul-Carime, N.Khelifa, J.P.Schermann. Phys.

Rev. Lett. 73, 1994, 2436.
4. F.Calogero. J.Math.Phys. 10, 1969, 2191.
5. A.M.Lomidze, Sh.M.Tsiklauri. Bulletin of the Georgian Academy of

Sciences, 159 1, 1999, 52.
6.Georgian Electronic Scientific Journal: Computer Sciences and Telecommunication 1-2002, 49.
7. A.M.Gorbatov, A.V.Bursak, Yu.N.Krilov, B.V.Rudak, Yad. Fiz. 40, 1984, 233; M.I.Haftel, V.B.Mandelzweig. Phys. Lett. A120, 1987, 232; Ann. Phys. 189, 1989, 29; Fabre de la Ripelle, Ann. Phys. 147, 1982, 281.
8. R.I.Jibuty, N.B.Krupennikova. The Hyperspherical Functions Method in Few - Body Quantum Mechanics. Tbilisi, 1984, (Russian).

## Tbilisi State University

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