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MODIFIED METHOD OF HYPERSPHERICAL FUNCTION FOR THREE-PARTICLE SYSTEM IN 3D SPACE WHEN PAIR INTERACTION BETWEEN PARTICLES IS $\left(a/\mathbf{r}^2 + \mathbf{b}/\mathbf{r}\right)$

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ABSTRACT. Three-particle system has been investigated by modified hyperspherical function method between particles by pair interaction $\left(a/\mathbf{r}^2 + \mathbf{b}/\mathbf{r}\right)$. For hyperradial wave function there has been obtained the system of infinitely coupled equations solved analytically by diagonal approximation. In the same approximation analytical expressions of binding energy have been obtained.

A number of physical phenomena can be described by singular potentials [1], and inverse square potential (critical singular potential) is especially interesting, because it can be used as an independent singular potential and power singular potentials [2] in a lot of physical spheres. In spite of this the research sphere is limited by: 1) Two particle system in $D \geq 1$ space; 2) Three and more particle systems in 1D space. Therefore the investigation of this three-particle system independently with critical singular potential and with its other power singular potential in 2D and 3D space extends knowledge and the sphere of its utilization and gives more precise definition about this potential. The results will be useful for the scientists, experts and specialists which are also working in the 2D space of development of science and technological spheres.

The problem of three-particles for pair inverse square interaction using modified hyperspherical function method has been studied in 2D space [3] and 3D space [4], respectively.

Two particles in correspondence with solution of Schordinger equation in 3D space when potential of interaction is the algebraic sum of square inverse potential and Coulomb potential, or when it is as follows:

$$\left(\frac{a}{r^2}+\frac{b}{r}\right) \tag{1}$$

has been studied in 3D space analytically [5], when b < 0.

The object of this work is to study theoretically three-particle system in 3D space. We use the modified method of hyperspherical function when pair interaction between particles is as in (1).

The aim of this paper is the following. In Section I, we discuss exact solution of Schrodinger equation for three-particle system in 3D space when pair interaction between particles is as (1). In Section II, we discuss the same system with same interaction when we used the modified method of hyperspherical function [3, 4]. In Section III, our main conclusions are summarized.

1. APPROXIMATE ANALYTICAL SOLUTION OF SCHRODINGER EQUATION FOR THREE-PARTICLE SYSTEM

Considering the three-particle system in 3D space which correspons to Schrodinger equation using hyperspherical function method is given in [4]. For Schrodinger equation we have got coupled differential equations system:

$$\left(\frac{\partial^2}{\partial\rho^2}-\left[\chi^2+\frac{(K+2)^2-0{,}25}{\rho^2}\right]\right)\chi_{RL}^{l_1l_2}(\rho)=$$
$$\sum_{K'l_1'l_2'M'} W_{KK'LL'MM'}^{l_1l_2l_1'l_2'}(\rho)\chi_{K'L'}^{l_1'l_2'}(\rho)\ , \tag{2}$$

where

$$\chi^2=-\frac{2\mu}{h^2}E\ , \tag{3}$$

and $W^{l_1l_2l'_1l'_2}_{KK'LL'MM'}(\rho)$ is defined by expression:

$$W^{l_1l_2l'_1l'_2}_{KK'LL'MM'}(\rho) =$$

$$= \frac{2\mu}{h^2}\left[J^{l_1l_2l'_1l'_2}_{KK'LL'MM'} + \sum_{\tilde{l}_1\tilde{l}_2} {}^{i}\langle \tilde{l}_1\tilde{l}_2 | l_1l_2\rangle^{j}_{KL}\, {}^{i}\langle \tilde{l}'_1\tilde{l}'_2 | l'_1l'_2\rangle^{j}_{KL}\, J^{\tilde{l}_1\tilde{l}_2\tilde{l}'_1\tilde{l}'_2}_{KK'LL'MM'} + \right.$$

$$\left. + \sum_{\tilde{\tilde{l}}_1\tilde{\tilde{l}}_2} {}^{i}\langle \tilde{\tilde{l}}_1\tilde{\tilde{l}}_2 | l_1l_2\rangle^{j}_{kL}\, {}^{i}\langle \tilde{\tilde{l}}'_1\tilde{\tilde{l}}'_2 | l'_1l'_2\rangle^{j}_{KL}\, J^{\tilde{\tilde{l}}_1\tilde{\tilde{l}}_2\tilde{\tilde{l}}'_1\tilde{\tilde{l}}'_2}_{KK'LL'MM'} \right]. \qquad (4)$$

In angular integral of (4) expression, e.g. for 1 and 2 particles are:

$$J^{l_1l_2l'_1l'_2}_{KK'LL'MM'} = \int \Phi^{*l_1l_2}_{KLM}(\Omega_i) U_{12} \Phi^{l'_1l'_2}_{K'L'M'}(\Omega_i) d\Omega, \qquad (5)$$

where $\Phi^{l_1l_2}_{KLM}(\Omega)$ are eigenvalues $K(K+4)$, of eigenfunction for square of K orbital moment, and create full collection of orthonormal basic functions that are following5

(5)

$$\Phi^{l_1l_2}_{KLM}(\Omega) = N^{l_1,l_2}_{K} \cos^{l_1}\alpha \sin^{l_2}\alpha\, P^{l_1+1/2,l_2+1/2}_{n}(\cos 2\alpha) \times$$

$$\times Y_{l_1m_1}(x) Y_{l_1m_1}(y), \qquad (6)$$

where

$$N^{l_1l_2}_{K} = \left[\frac{2n!(K+2)\Gamma(l_1+l_2+2+n)}{\Gamma(n+l_1+3/2)\Gamma(n+l_2+3/2)} \right]^{1/2}$$

and

$$n = \frac{K - l_1 - l_2}{2}. \qquad (7)$$

l_1 და l_2 are orbital moments of Jacoby coordinates.

We consider only one equation from equation's system (2), when $K = K'$.

Total potential of system that is between particles by pair interaction is:

$$U_{123}(x,y)=a_{12}r_{12}^{-2}+a_{13}r_{13}^{-2}+a_{23}r_{23}^{-2}+b_{12}r_{12}^{-1}+b_{13}r_{13}^{-1}+b_{23}r_{23}^{-1} \quad (8)$$

(8)

Solution of Schrodinger equation for the same particle system (2) when it is in extension angular integral (see (5)) and terms of integral, for example 1 and 2, will be:

$$J_{12}=\int \Phi^{*}_{K}(\Omega)\left(a_{12}r_{12}^{-2}+b_{12}r_{12}^{-1}\right)\Phi_{K}(\Omega)d\Omega \qquad (9)$$

and expression (9) is solved analytically.

With account of obtained results and zero approximation (2), solution of Schrodinger equation has been gotten as:

$$\left(\frac{\partial^2}{\partial\rho^2}-\left[\chi^2+\frac{(K+2)^2-.25}{\rho^2}\right]\right)\chi_K(\rho)=\left(\frac{1}{\rho^2}W_1+\frac{1}{\rho}W\right)\chi_K(\rho), \quad (10)$$

where

$$\left(\frac{1}{\rho^2}W_1+\frac{1}{\rho}W\right)=\left[J_{12}+\langle 00\,|\,00\rangle_2\langle 00\,|\,00\rangle_2 J_{13}+\right.$$

$$\left.+\langle 00\,|\,00\rangle_2\langle 00\,|\,00\rangle_2 J_{23}\right]. \qquad (11)$$

After the solution of Schrodinger equation (10) for energetical levels of the system we have:

$$E=-\frac{\hbar^2}{2m}\left(\frac{W}{2N+2\lambda}\right)^2 \qquad (12)$$

and for radial components of wave function it is:

$$\varphi(\rho)=(2\chi)^{\lambda+1/2}\left[\frac{\Gamma(N+1)}{2\lambda}\right]^{1/2}\frac{1}{[\Gamma(N+2\lambda)]^{3/2}}\rho^{(\lambda-1/2)}\times$$

$$\times\exp(-\chi\rho)\,L^{2\lambda-1}_{N+2\lambda-1}(2\chi\rho),\qquad(13)$$

where

$$\lambda=\frac{1}{2}+\sqrt{(K+2)^2+W_1}\ ;\ -N=\lambda+W/(2\chi).\qquad(14)$$

2. SOLUTION OF SCHRODINGER EQUATION FOR THREE-PARTICLE SYSTEM USING THE MODIFIED METHOD OF HYPERSPHERICAL FUNCTION WHEN PAIR INTERACTION BETWEEN PARTICLES IN 3D SPACE IS AS (1)

When K = K'= 0 we have expression (10). For solution of this equation we can use modified method of hypersperical function [3].

The main idea of the MMHF is that wave function Ψ represents the product of two functions, where the first is the main hyperspherical function and the second is the correlation function- $\zeta=\exp(f)$ that is defined by singularity and clustering properties of the wave function and it is equal to

$$f=-\sum_{i=1}^{3}\gamma_i r_i,\qquad(15)$$

where r_i is the distance between the particles and γ_i is determined according to physical considerations.

Taking into account the relation between three different sets of Jacobi coordinates [7], the expression (15) can be presented as following:

$$\sum_{i=1}^{3}\gamma_i z_i=\rho(G_1\cos\alpha+G_2\sin\alpha),\qquad(16)$$

where: $G_1=\gamma_1+\gamma_2\cos(\phi_{23}+\phi_{31})-\gamma_3\cos\phi_{31};$

$$G_2=\gamma_2\sin(\phi_{23}+\phi_{31})-\gamma_3\sin\phi_{31}.\qquad(17)$$

If we substitute expression (15) into (10), and after transformation by hyperradial differential equation we obtain:

$$\left(\frac{\partial^2}{\partial\rho^2}+\left(\frac{5}{\rho}-W_2'\right)\frac{\partial}{\partial\rho}+\frac{W_4'+W_3'}{\rho}-\frac{2\mu}{{}^2}\frac{J}{\rho}+\right.$$
$$\left.+\left(\chi^2+W_6'\right)-\frac{2\mu}{h^2}\frac{K(K+4)+J_1}{\rho^2}\right)\psi(\rho)=0, \qquad (18)$$

where:

$$W_2'=(G_1-G_2)*\frac{4}{15}; \qquad W_4'=3W_2';$$
$$W_6{}'=G_1^2+G_2^2; \qquad W_3'=G_1\left(\frac{4}{15}-\frac{3\pi}{8}\right)-\frac{42}{105}G_2; \qquad (19)$$

J_1 is the sum of the first members and J is the sum of second members in expression (9) (and same expressions for particles 13 and 23).

Taking into account the asymptotic behavior of the equation (18), let us seek a solution as the following:

$$\psi(\rho)=\exp(-\delta\rho)\rho^{\sigma}\varphi(\rho), \qquad (20)$$

where:

$$\sigma=-1+\sqrt{9+\frac{2\mu}{r^2}J_1}; \qquad \delta=\frac{\sqrt{W_2'^2-4\left(\chi^2+W_6'\right)}-W_2'}{4}. \qquad (21)$$

When substituting expression (20) into the equation (18), then for $\varphi(\rho)$ we obtain the equation of hypergeometrical function:

$$\left(r\frac{\partial^2}{\partial\rho^2}+(-r+2\sigma+3)\frac{\partial}{\partial\rho}+\frac{3W_2'+W_3'-3\delta-V_C}{2\delta+W_2'}-\sigma\right)\varphi(\rho)=0, \qquad (22)$$

where:

$$r = \left(2\delta + W_2'\right)\rho , \tag{23}$$

and

$$V_C = \frac{2\mu}{{}^{2}} J . \tag{24}$$

If we take into account that three-body system is bonded then solution of the (8) equation is represented as the following type of hypergeometrical function:

$$\varphi\left(\rho\right) = C_1 F(a, b, r) , \tag{25}$$

where:
$$b = \sigma + 5;$$

$$a = \frac{(W_4' + W_3' - 2\delta\sigma - 5\delta - \sigma W_2' - V_c)}{2\delta + W_2'} = -N ; \quad N = 1,2.... \tag{26}$$

For binding energy we received:

$$E_N = -\frac{{}^{2}}{8\mu}\left\{\left[\left(\frac{4\left(W_4' + W_3' - V_c\right) - W_2'(2\sigma + 5 - 3N)}{2\sigma + 5 - N}\right)^2 - W_2^{\prime 2}\right] + 4W_6'\right\} \tag{27}$$

3. CONCLUSIONS AND ANALYSES

The dependence of binding energy of three body system on the global quantum number N obtained as a solution results in the expressions (12) and (27) that is given in the Table 1.

Table 1. Dependence of the binding energy of the three-body system in 3D space on the global quantum number N

N	Binding energy -E (*a*.u.), by expression (12)	Binding energy -E (*a*.u.), by expression (27)
0	0.3156439	0.304168
1	0.1264723	0.123532

2	0.067679	0.066523
3	0.042062	0.041494
4	0.0286507	0.028331
5	0.0207633	0.020566
6	0.0157354	0.015605
7	0.0123351	0.012244
8	0.0099287	0.009863

(In these results we assume that correlation parameters are the same for all particles. Interaction constants are the same for all particles too and satisfy the conditions: $a_{ij} < 0$, and $b_{12} = b_{13} < 0$, $b_{23} > 0$. If the conditions (14) and (21) are satisfied their variation doesn't give any qualitatively new results).

The Table shows that the binding energy of three-body system depending on the global quantum number N calculated from (12) and (27) formulae is exactly the same.

We can say that the use of MMHF doesn't give any quantity variation.

If we compare the MMHF with the results of $1/r^2$ potential, we can say that the MMHF for (1) potential doesn't give qualitative variation, because the potential contains 1/r.

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